# Adaptive Linearization Schemes for Weakly Nonlinear Systems Using Adaptive Linear and Nonlinear FIR Filters

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**Abstract**

Three adaptive linearization schemes are proposed in this paper. In the first scheme, linearization is performed by canceling nonlinearity at the output of a physical system. In the second scheme, a nonlinear post-processor is employed to post-distort signals, while in the third scheme, a pre-processor is used. In these schemes, necessary estimates of linear and nonlinear operators are provided by adaptive linear and nonlinear filters. Encouraging results have been observed in the simulations.

Key Words: Adaptive Linearization, Adaptive Linear and Nonlinear Filters.

## 1. Introduction

System linearity is desired in many applications where nonlinearities exist. Thus it is necessary to cancel out or reduce the nonlinearities in the system. Following are some applications where linearization should be employed:

*) In integrated filters, a resistor should be replaced by a transistor, which has reasonably good linearity at small signals, to fully integrate continuous-time filters. However, a transistor presents intolerable nonlinearity at large signal swings [8][9].

*) In optical communication, distortions caused by the nonlinearities in the analog drive circuitry and LED or laser should be reduced to satisfy distortion requirements [10].

*) A loudspeaker has several major sources of nonlinearity, including non-uniform magnetic field and nonlinear suspension system [6][7]. Nonlinear distortion is often a few percent of the output signal, and it is very desirable to reduce it.

*) A critical issue in bandwidth-efficient QAM in digital microwave radio systems is nonlinearity of the high-power amplifier in a satellite. Adaptive predistortion techniques have been proposed to compensate the nonlinear distortion [11][12]

There are some drawbacks with the existing linearization approaches. Most of the linearization methods for integrated continuous-time filters require device matching. This matching can only be satisfied to a certain degree because of manufacturing fluctuations. Feedback technique has difficulties linearizing a loudspeaker system because of a delay involved in the feedback path. Most of the existing methods, except for some of those for linearization of satellite channels, rely on fixed circuits or devices, thus their performance will be degraded by aging, temperature, and an ever-changing environment.

Adaptive approaches may provide a good solution for some applications. An approach for adaptive linearization was proposed in [1] for systems described by state-space equations. However, it is only applicable to the class of systems where cancellation of the internal feedback can be performed physically. This paper introduces three new adaptive linearization schemes, and each of them may have its own applications.

## 2. Adaptive FIR Filters and Volterra Series

This section reviews the principles of adaptive linear and nonlinear FIR filters and the Volterra series. These principles will be used later in this paper. An adaptive linear FIR filter has the following form [2]

$$y(k)=\sum_{i=0}^{n}h(i)u(k-i) \quad (1)$$

where $n$ is the filter order, $u$ the input signal, $k$ the discretized time, $h$ the impulse response, or coefficients of the filter.

Using the LMS algorithm, we can update the coefficients according to [2]

$$h^{k+1}(i)=h^{k}(i)+2\mu e(k)u(k-i) \quad (2)$$

where $\mu$ is the step size and $e$ is the error, defined as the difference between the reference signal and the filter output.

The Volterra series forms the basis for adaptive nonlinear FIR filters and the adaptive linearization schemes proposed in this paper. For a nonlinear system satisfying certain conditions, the output $y(k)$ can be expanded into a Volterra series

$$y(k)=\sum_{i_1=0}^{\infty}h_1(i_1)u(k-i_1)+\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}h_2(i_1,i_2)u(k-i_1)u(k-i_2)+\cdots$$
$$+\sum_{i_1=0}^{\infty}\cdots\sum_{i_m=0}^{\infty}h_m(i_1,i_2,\cdots,i_m)u(k-i_1)u(k-i_2)\cdots u(k-i_m)+\cdots \quad (3)$$

where $h_1, h_2, \ldots, h_m, \ldots$, are the coefficients of the system. The sum with $h_1$ can be considered as a convolution of the input signal, $u(k)$, with the impulse response, $h_1(k)$, of a linear system. This sum is a linear term. The sum with $h_i$ will be referred to as *ith* power term, since if the input $u(k)$ is multiplied by a scalar $\alpha$, this term will yield a factor $\alpha^i$. Particularly, the sum with $h_2$ and the sum with $h_3$ will be referred to as quadratic term and cubic term, respectively. The linear term models the linearity of the system, and the rest of the series together models the nonlinearity.

Generally speaking, the series has an infinite number of terms and each term is an infinite sum. The computation and memory requirements make it impossible to base adaptive filters on this series if no simplification is made. In practice, the series is able to model many system reasonably well if it just contains the several major terms and each term is a finite sum. Thus, we can employ the truncated series to construct adaptive filters. Furthermore, we can consider that $h_2(i_1,i_2)$, $h_3(i_1,i_2,i_3)$, and $h_m(i_1,i_2,\cdots,i_m)$ are symmetric, namely, the indexes of $h_2(i_1,i_2)$, $h_3(i_1,i_2,i_3)$, or $h_m(i_1,i_2,\cdots,i_m)$ are exchangeable. Then, an adaptive nonlinear filter can be based on the following truncated series [3][4][5]

$$y(k)=\sum_{i_1=0}^{n_1}h_1(i_1)u(k-i_1)+\sum_{i_1=0}^{n_2}\sum_{i_2=i_1}^{n_2}h_2(i_1,i_2)u(k-i_1)u(k-i_2)+\cdots$$
$$+\sum_{i_1=0}^{n_m}\sum_{i_2=i_1}^{n_m}\cdots\sum_{i_m=i_{m-1}}^{n_m}h_m(i_1,i_2,\cdots,i_m)u(k-i_1)u(k-i_2)\cdots u(k-i_m) \quad (4)$$

where $m$ is the total number of terms in the filter, $n_1$ is called the order of the linear term, similarly, $n_2, n_3, ...,$ and $n_m$ are called orders of the nonlinear terms. Note the changes in the upper and lower limits of the summations. Obviously, this filter has a finite impulse response (FIR), thus, the name adaptive nonlinear FIR filter.

Updating of the coefficients of the linear term of an adaptive nonlinear FIR filter is performed according to (2), and updating of other coefficients is done as follows [3][4][5]

$$h_j^{k+1}(i_1,i_2,\cdots,i_j) = h_j^k(i_1,i_2,\cdots,i_j) + 2\mu e(k)u(k-i_1)u(k-i_2)\cdots u(k-i_j) \quad (5)$$

where $j = 2, 3, ..., m$ and $\mu_j$ is the step size for the $jth$ power term.

## 3. Adaptive Linearization Schemes

Three adaptive linearization schemes will be discussed in this section. These schemes are based on the Volterra series and adaptive FIR filters discussed above.

### Linearization by Cancellation at the Output

As discussed in the above section, the Volterra series represents a nonlinear system by two subsystems, one is purely linear and another is purely nonlinear. This is described notationally by

$$y_p(k) = y_{L_p}(k) + y_{N_p}(k)$$

$$= L_p(u(k)) + N_p(u(k)) \quad (6)$$

where $y_{L_p}$ is the output of the linear subsystem with linear operator $L_p$, and $y_{N_p}$ is the output of the purely nonlinear subsystem with nonlinear operator $N_p$. In this paper, whenever it is necessary to distinguish the variables of a physical system from those of an adaptive filter, we use the subscript $p$ for the variables of the physical system.

It is obvious that we can linearize a nonlinear system by subtracting an estimate of the output of the purely nonlinear subsystem from the output of the physical system. The estimate of the output of the purely nonlinear subsystem can be obtained from an adaptive nonlinear filter. This adaptive linearization scheme is shown in Fig.1.

However, for some applications, such as a loudspeaker system, it is hard to perform signal subtraction at the output side of a system. In some cases, it is desired or necessary to pre-distort a signal at the input side of a system, while in other cases, post-distorting of signals may be required.

### Linearization Using A Post-Processor

For some applications, for example, those in communications systems, signals can only be processed after being received. A post-processor can be applied to linearize such a system, as shown in Fig.2. One method is proposed here for a weakly nonlinear system.

In the following discussion, inversion modeling of the linear behavior of a nonlinear system will be used. Let $L^{-1}$ indicate the linear operator obtained by an adaptive linear filter which performs reverse modeling of a physical system described by (6). Then, we can have $L^{-1}$, satisfying

$$L^{-1}L_p = z^{-\delta} \quad (7)$$

where $z^{-\delta}$ indicates a delay of $\delta$ samples and $\delta$ usually must be nonzero so that the adaptive filter can converge. If the nonlinearity of a physical system is weak, a post-processor with output

$$y(k) = y_p(k-\delta) - N(L^{-1}(y_p(k))) \quad (8)$$

can reduce (though not eliminate) the nonlinear distortion, thus, linearizing the system. The notation $N$ indicates an estimate of the operator $N_p$. We can verify this idea by some simple algebraic manipulations. The delayed output of the physical system can be written as

$$y_p(k-\delta) = L_p(u(k-\delta)) + N_p(u(k-\delta))$$

Then the output of the nonlinear post-processor is

$$y(k) = L_p(u(k-\delta)) + N_p(u(k-\delta)) - N(L^{-1}(L_p(u(k)) + N_p(u(k))))$$

$$= L_p(u(k-\delta)) + N_p(u(k-\delta)) - N(u(k-\delta) + L^{-1}(N_p(u(k))))$$

where (7) is used. Assuming that the nonlinearity is weak, namely

$$|L_p(u(k))| \gg |N_p(u(k))|$$

we have

$$|u(k-\delta)| \gg |L^{-1}(N_p(u(k)))| \quad (9)$$

where (7) is again employed. Then, we have

$$y(k) \doteq L_p(u(k-\delta)) + N_p(u(k-\delta)) - N(u(k-\delta))$$

$$\doteq L_p(u(k-\delta)) \quad (10)$$

The remaining nonlinearity in the output is of higher order and the output of the processor is the linearized output $y_{ld}$, where the subscript $ld$ stands for linearized.

To implement the linearization schemes, the operators $L^{-1}$ and $N$ are needed. Adaptive linear and nonlinear FIR filters can be used to provide their estimates. The adaptive implementation using adaptive FIR filters is shown in Fig.3. The adaptive nonlinear FIR filter models the "forward" behavior of the physical system and gives the operators $L$ and $N$, which are the estimates of $L_p$ and $N_p$. The adaptive linear FIR filter models the "reverse" behavior of the linear part of the physical system and gives the operator $L^{-1}$, the estimate of $L_p^{-1}$ with a difference of a delay operator. The input of the adaptive linear filter can be either the output of the physical system or the output of the linear subsystem of the adaptive nonlinear filter (see dashed lines in Fig.3). The linear FIR filter of the processor is copied from the adaptive linear FIR filter, and the purely nonlinear FIR filter of the processor is a copy of the nonlinear operator $N$ of the adaptive nonlinear filter. The copying could be done right after the start of the adaptation process. However, at the beginning, the adaptive filters do not have good estimates, thus, the nonlinear post-processor may not reduce the nonlinear distortion and may even worsen it. Therefore, it is better to perform the copying after the adaptive filters get reasonably good estimates. If the input of the adaptive linear filter is the output of the physical system, then, Fig.3. can be easily modified so that the adaptive linear filter can serve as the linear filter of the processor and computation can be reduced.

### Linearization Using A Pre-Processor

For other linearization applications, a nonlinear processor is needed to pre-distort signals, as shown in Fig.4. A nonlinear processor with the following nonlinear mapping

$$y_i(k) = u(k-\delta) - L^{-1}(N(u)) \quad (11)$$

can perform the task. This can be verified easily. The output of the physical system is

$$y_p(k) = L_p(y_i(k)) + N_p(y_i(k))$$

$$= L_p(u(k-\delta) - L^{-1}(N(u(k)))) + N_p(u(k-\delta) - L^{-1}(N(u(k))))$$

$$\doteq L_p(u(k-\delta)) \quad (12)$$

where (7) and (9) are used. Hence, the output of the physical system is the linearized output, namely, $y_p = y_{ld}$.

This scheme can also be implemented using adaptive filters, as shown in Fig.5. The input of the adaptive linear filter is either the output of the physical system or the output of the linear subsystem of the adaptive nonlinear filter. The linear FIR filter is copied from the adaptive linear FIR filter and the purely nonlinear FIR filter is copied from the nonlinear part of the adaptive nonlinear FIR filter. As in the case of linearization using a post-processor, it is better to copy after the adaptive filters have run for some time and have good estimates.

## 4. Simulation Results

Numerical experiments on several different systems have been performed to test the adaptive linearization schemes. Similar results have been obtained and typical results are presented in this section.

In the following tests, the physical system was modeled by a Volterra series with a linear term, a quadratic term and a cubic term, the adaptive nonlinear filter had the same orders as the physical system, and initially, all the coefficients of the adaptive filters were set to zero. For the scheme of cancellation at the output, the output of the nonlinear part of the physical system was the original distortion. The residual distortion after linearization was the difference between the output of the linear part of the physical system and the linearized output which was the physical system output subtracted by the output of the nonlinear part of the adaptive filter. To measure the residual distortion for other two schemes, we have used a reference system which was a copy of the physical system. Its input was $u(k-\delta)$, a delayed version of the original input signal since the linearized signal was delayed by this amount in these two schemes. The output of the linear part of the reference system is the ideal linear output, then the residual distortion was measured as the difference between this linear signal and the linearized signal.

### Test 1

The physical system had the orders $n_{p1}=10$, $n_{p2}=3$, and $n_{p3}=2$. The linear part was

$$y_{Lp}(k)=0.2u(k)+0.5u(k-1)+0.3u(k-2)+1.2u(k-3)+0.7u(k-4)+$$

$$0.05u(k-5)+0.01u(k-6)+0.01u(k-7)+0.01u(k-8)$$

$$-0.008u(k-9)-0.005u(k-10)$$

The quadratic term was

$$y_{quadratic}(k)=0.01u^2(k)-0.001u(k)u(k-1)-0.001u(k)u(k-2)$$

$$+0.008u(k)u(k-3)+0.011u^2(k-1)+0.003u(k-1)u(k-2)+$$

$$0.001u(k-1)u(k-3)+0.009u^2(k-2)+\ 0.002u(k-2)u(k-3)+$$

$$0.008u^2(k-3)$$

and the cubic term was

$$y_{cubic}=0.005u^3(k)+0.003u^2(k)u(k-1)-0.005u^2(k)u(k-2)+$$

$$0.009u(k)u^2(k-1)-0.006u(k)u(k-1)u(k-2)-0.007u(k)u^2(k-2)+$$

$$0.008u^3(k-1)-0.001u^2(k-1)u(k-2)+0.002u(k-1)u^2(k-2)+$$

$$0.001u^3(k-2)$$

The order of the adaptive linear filter was chosen as $n=50$. The step sizes were 0.005 for $h_1$ of the adaptive nonlinear filter, and 0.001 for $h_2$, $h_3$ of the adaptive nonlinear filter and $h$ of the adaptive linear filter. The mean square (MS) value of original nonlinear distortion of the physical system was $-24.1dB$. The MS value of the linear signal, namely, the non-distorted signal, was $2.9dB$.

The reduction in distortion versus the number of iterations is shown in Fig.6 for the scheme of linearization by cancellation at the output. This curve shows that at $6k$ iterations the distortion was reduced to $-107dB$, and after $20k$ iterations the distortion was reduced to $-290dB$. In this case, subtraction of the nonlinear output was performed once the adaptive filter started to work. The output distortion was actually worse than the original distortion for the first $1k$ iterations due to transients. In some applications, it is necessary to avoid this, and so this subtraction should be performed after the adaptive filter gets better estimates. The performance of the adaptive nonlinear filter can also be deduced from this figure since the curve of the MS error for forward identification of the physical system had a difference of just a few $dB$ with the curve shown in Fig.6 and had a very similar shape. For the scheme with a nonlinear post-processor and the scheme with a pre-processor, the distortions have been reduced to $-53.1dB$ and $-46.6dB$, respectively, from $-24.1dB$. The test demonstrates that the scheme of cancellation at output can be essentially perfect if the adaptive filter is able to identify the physical system well, while other two techniques have certain distortion residuals.

### Test 2

The orders of the physical system in Tests 2 and 3 were made larger than those in the Test 1 and were $n_{p1}=20$, $n_{p2}=10$, and $n_{p3}=3$. For brevity, the values of the coefficients will not be listed here. The order of the adaptive linear filter was chosen as $n=150$. The step sizes were 0.001 for $h_1$, $h_2$ of the adaptive nonlinear filter and $h$ of the adaptive linear filter, 0.0005 for $h_3$. The MS values of original nonlinear distortion and linear signal of the physical system were $-27dB$ and $2.6dB$, respectively. The distortion has been reduced to $-290dB$, $-58.5$, and $-48.0dB$ by the scheme of cancellation at the output, the scheme with a post-processor, and the scheme with a pre-processor, respectively.

### Test 3

All the conditions in both Test 2 and Test 3 were the same, except for the nonlinear parts. The nonlinear coefficients of the physical system in Test 3 were chosen so that there was a lower distortion in the original physical system than that of Test 2. At $40k$ iterations, the distortion was reduced to $-289dB$ for the scheme of linearization by cancellation at the output, to $-81dB$ for the scheme with a post-processor, and to $-77dB$ for the scheme with a pre-processor from the original distortion of $-44dB$. The distortion reductions in Test 3 for the schemes with a pre-processor and a post-processor were larger than those in Test 2 since the original distortion in Test 3 was smaller, satisfying the assumption in (9) better.

In all tests, no significant differences in the results were observed whether the output of the physical system or the output of the linear part of the adaptive nonlinear filter was used as the input signal to the adaptive linear filter.

## 5. Summary

Three new adaptive linearization schemes have been developed. The schemes are attractive in that the resultant systems are not complicated, making it easy to implement in both hardware and software. The scheme using a post-processor and the scheme using a pre-processor are designed for weakly nonlinear systems, while the scheme of linearization by cancellation at the output can be applied to problems with stronger nonlinearities. These methods can find applications in acoustical systems, communications systems, etc. We are currently investigating application of the scheme with a pre-processor to linearization of a complete model of a loudspeaker system and application of the scheme with a post-processor to equalization of a nonlinear data communication channel.

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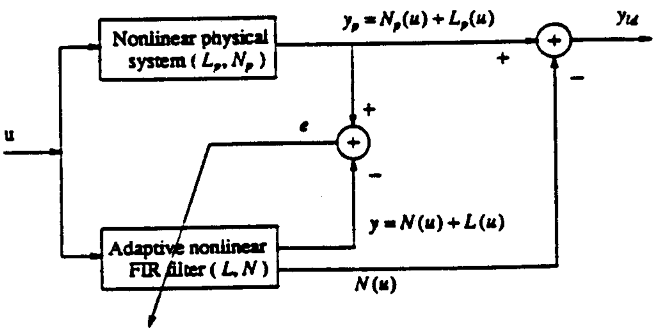


Fig.1 Adaptive linearization by canceling the nonlinearity at the output.

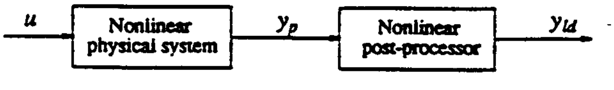


Fig.2 A nonlinear post-processor is placed at the OUTPUT side of the nonlinear physical system to POST-distort the signal.

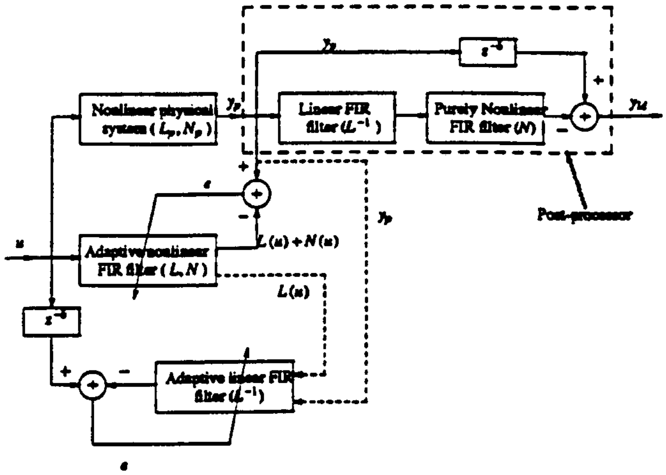


Fig.3 Adaptive implementation using FIR filters for the scheme with a post-processor. Either one of the two dashed lines could be used. The linear filter is copied from the adaptive linear FIR filter, and the nonlinear filter N is a copy of the nonlinear part of the adaptive nonlinear FIR filter.

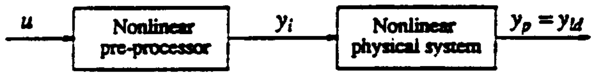


Fig.4 A nonlinear pre-processor is placed at the INPUT side of the nonlinear physical system to PRE-distort the signal.

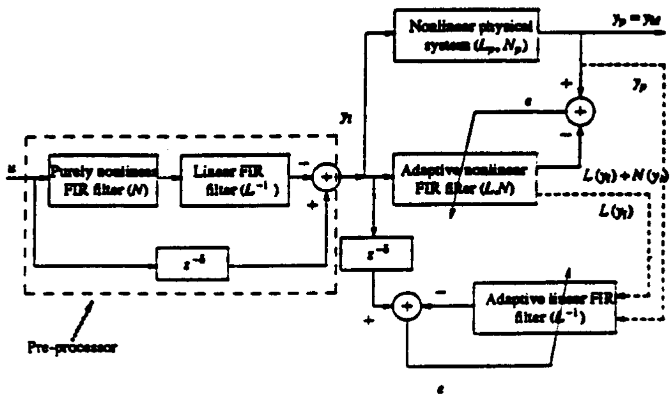


Fig.5 Adaptive implementation using FIR filters for the scheme with a pre-processor. Either one of the two dashed lines could be used. The linear FIR filter is copied from the adaptive linear FIR filter, and the nonlinear filter N is a copy of the nonlinear part of the adaptive nonlinear FIR filter.

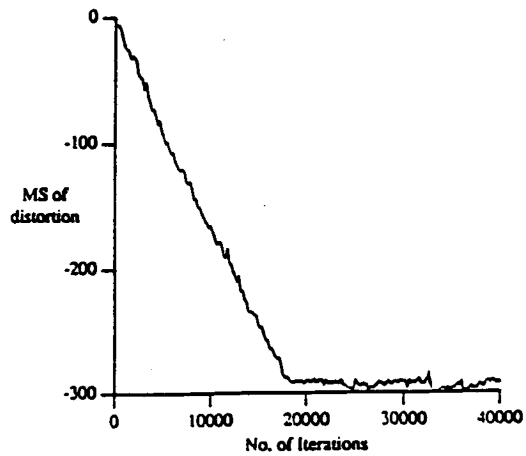


Fig.6 Reduction in distortion for linearization by cancellation at the output in Test 1.