Notes for 1991 ICIAM:

Stability in a Sigma-Delta Modulator

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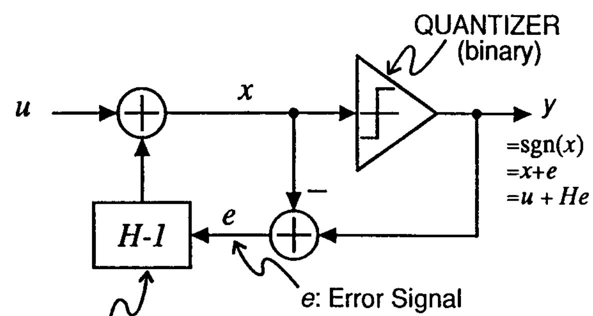
ABSTRACT

Sigma-delta modulators are used in today's best analog-to-digital converters.

These circuits are almost entirely linear: a single signum nonlinearity is the only nonlinear element.

Despite their widespread use, the question "Under what conditions is a given modulator stable?" does not have a satisfactory answer.

A SIMPLIFIED MODULATOR



H: Error Transfer Function.
$H-1$ must be strictly causal, i.e. $h_0 = 1$.

ITS DESCRIBING EQUATIONS

Time-Domain	Z-Transform Domain
$x_n = u_n + \sum_{i=1}^{\infty} h_i e_{n-i}$	$X = U + (H-1)E$
$y_n = \mathrm{sgn}(x_n)$	$Y = X + E$
$e_n = y_n - x_n$	$\Rightarrow Y = U + HE$

The time-domain equations form a set of nonlinear recursion equations.

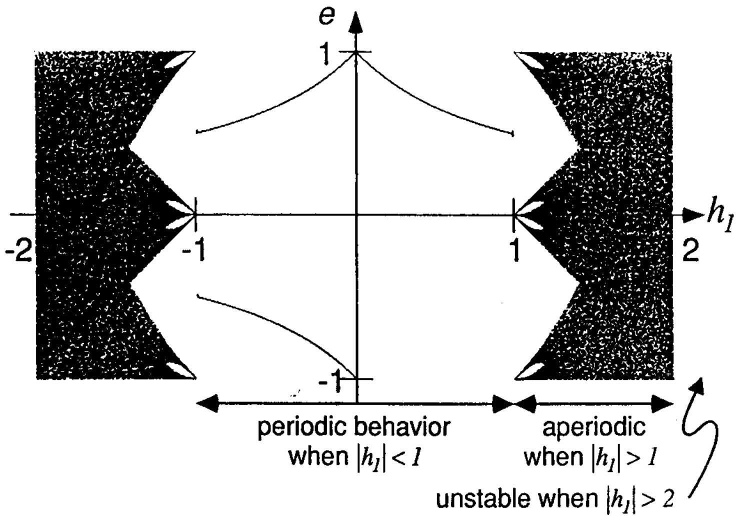
1-DIMENSIONAL CASE, u=0

The recursion equations are

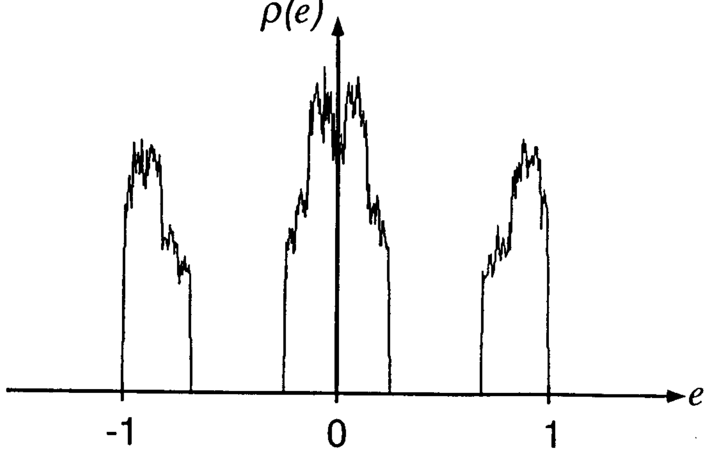
$$x_n = h_1 e_{n-1}$$
$$e_n = sgn(x_n) - x_n$$

What is the behavior of the e sequence for various values of h_1?

STEADY-STATE RANGE OF e



HISTOGRAM OF e FOR h_1=1.25



WHAT WE UNDERSTAND:

The periodic behavior.

The reason for the aperiodic behavior.

The limits on both e and the stable range of h_1.

WHAT WE DON'T:

The details of the distribution of e.

Is the aperiodic behaviour chaos?

2-DIMENSIONAL CASE, u=0

The recursion equations are

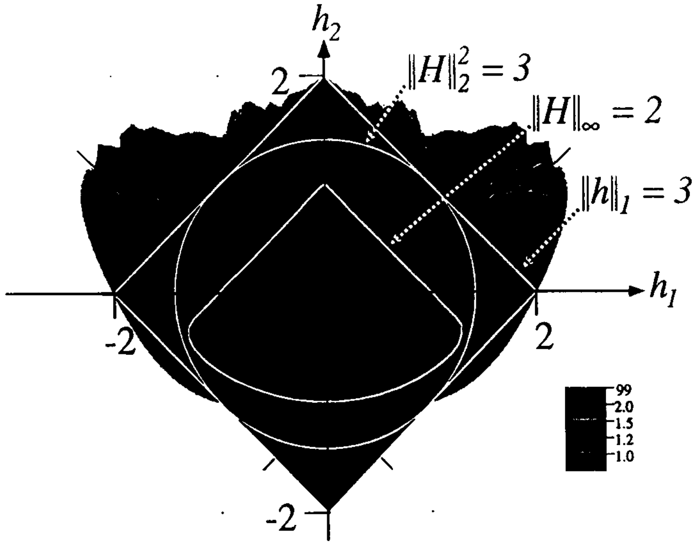
$$x_n = h_1 e_{n-1} + h_2 e_{n-2}$$

$$e_n = sgn(x_n) - x_n$$

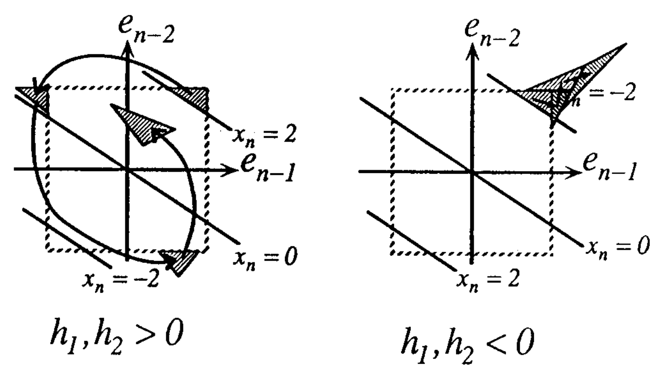
Now what is the behaviour of e for various values of h_1 and h_2?

In particular, under what conditions is e bounded?

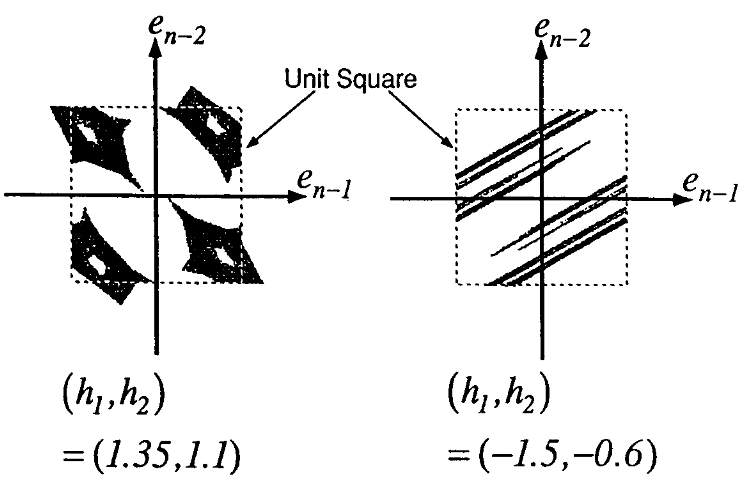
$\|e\|_\infty$ VS (h_1, h_2)



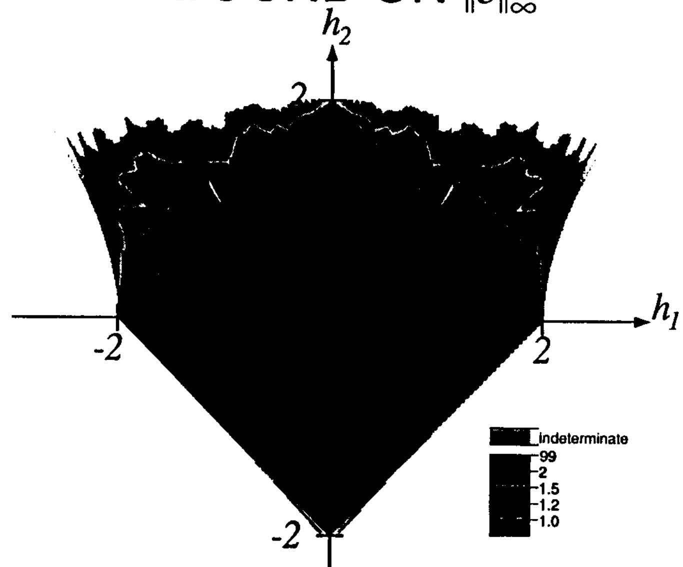
FOLLOWING THE CRITICAL CORNER IN STATE-SPACE



RANGE OF (e_{n-1}, e_{n-2})



A BOUND ON $\|e\|_\infty$



WHAT WE "UNDERSTAND":

How stability works for the majority of cases when the state can range over the whole unit square.

WHAT WE DON'T:

The range of state-space that is accessible. Can it be fractal?

Necessary and sufficient conditions for zero-input stability in the second-order case.

GENERAL CASE: $u \neq 0$

If $|e_i| \leq 1$ for $i < n$,

$$\begin{aligned} |x_n| &= \left| u_n + \sum_{i=1}^{\infty} h_i e_{n-i} \right| \\ &\leq |u_n| + \sum_{i=1}^{\infty} |h_i e_{n-i}| \\ &\leq \|u\|_\infty + \|h\|_1 - 1 \end{aligned}$$

If $\|h\|_1 \leq 3 - \|u\|_\infty$ then $|x_n| \leq 2$ and $|e_n| \leq 1$.

Thus $\|h\|_1 \leq 3 - \|u\|_\infty \Rightarrow \|e\|_\infty = 1$.

CONCLUSIONS

The set of zero-input stable modulators is quite complicated. A simple rule is unlikely to be able to produce this set.

A sufficient test for 1-stability is easily proven, but is too restrictive.